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$gl_{q,q^*}(n)$-Covariant Multimode Oscillators and q-Symmetric States

W.-S. Chung[1] **and S.-K. Lee**[1]

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In this paper the $ql_q(n)$ oscillator algebra is extended to the complex deformation parameter case [$gl_{q,q^*}(n)$ algebra], and q-symmetric states for $gl_{q,q^*}(n)$-covariant multimode oscillator system are investigated.

The q-deformed Lie algebra implies some specific deformations of classical Lie algebras. The first q-deformation of the classical Lie algebra $su(2)$ was constructed by Jimbo (1985, 1986) and Drinfeld (1985).

The q-deformation of Heisenberg algebra was made by Arik and Coon (1976), (1989), Macfarlane and Biedenharn (1989). Recently there has been some interest in more general deformations involving arbitrary real functions of weight generators and including q-deformed algebras as a special case (Polychronakos, 1990; Rocek, 1991; Daskaloyannis, 1991; Chung *et al.*, 1993; Chung, 1994).

Recently Greenberg (1991) studied the following q-deformation of multimode boson algebra:

$$a_n a_j^\dagger - q a_j^\dagger a_i = \delta_{ij}$$

where the deformation parameter q has to be real. The main problem of Greenberg's approach is that we cannot derive the relation among the a_i operators at all. Moreover, the above algebra is not covariant under $gl_q(n)$ algebra. In order to solve this problem some physicists found the q-deformed multimode oscillator algebra which is covariant under $gl_q(n)$ algebra (Jagannathan *et al.*, 1992; Pusz and Woronowicz, 1989) when the deformation parameter q is a real number.

[1] Department of Physics and Research Institute of Natural Science, Gyeongsang National University, Jinju, 660-701, Korea.

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For real deformation parameter, the Pusz and Woronowicz (1989) $gl_q(n)$-covariant multimode oscillator algebra satisfies the commutation relation

$$
\begin{aligned}
a_i a_j &= q^{-1} a_j a_i \qquad (i < j)\\
a_i^\dagger a_j^\dagger &= q a_j^\dagger a_i^\dagger \qquad (i < j)\\
a_i a_j^\dagger &= q a_j^\dagger a_i^\dagger \qquad (i \neq j)\\
a_i a_i^\dagger &= 1 + q^2 a_i^\dagger a_i + (q^2 - 1) \sum_{k=i+1}^{n} a_k^\dagger a_k
\end{aligned} \tag{1}
$$

where $a_i^\dagger$ is a Hermitian conjugate of a_i only when the deformation parameter q is real.

In this paper we extend (1) to the arbitrary complex deformation parameter case. In order to do so, we consider the R-matrix formulation of q-oscillator algebra as follows:

$$a_i a_j = R^{ij}_{kl} a_l a_k \tag{2}$$

where the R-matrix is defined as

$$
\begin{aligned}
R &= R^{ij}_{kl} e_{ik} \otimes e_{jl}\\
&= \sum_i e_{ii} \otimes e_{ii} + q \sum_{i \neq j} e_{ii} \otimes e_{jj} + (1 - q^2) \sum_{i > j} e_{ij} \otimes e_{ji}
\end{aligned} \tag{3}
$$

Here e_{ij} is the $n \times n$ matrix with entry 1 at position (i, j) and 0 elsewhere and R is an $n^2 \times n^2$ matrix. Using (3), the relation (2) becomes

$$a_i a_j = q^{-1} a_j a_i \qquad (i < j) \tag{4}$$

If we assume q to be an arbitrary complex number, we can obtain from (4)

$$a_i^\dagger a_j^\dagger = q^* a_j^\dagger a_i^\dagger \qquad (i < j) \tag{5}$$

The associativity for the product $a_i\, a_j a_k$ and $a_i^\dagger a_j^\dagger a_k^\dagger$ is satisfied from the Yang–Baxter equation for the R-matrix. Now we have to find the commutation relation for a_i and $a_j^\dagger$. Let us assume that they take the form

$$a_i a_j^\dagger = \delta_{ij} + C^{ijij}_{kl} a_k^\dagger a_l \tag{6}$$

The concrete form of the matrix C^{ij}_{kl} is fixed by demanding the associativity of the product $a_i a_j a_k^\dagger$ and $a_i a_j^\dagger a_k^\dagger$.

Then (6) is written in the form

$$
\begin{aligned}
a_i a_j^\dagger &= q a_j^\dagger a_i \qquad (i < j)\\
a_i a_j^\dagger &= q^* a_j^\dagger a_i \qquad (i > j)\\
a_i a_i^\dagger &= 1 + |q|^2 a_i^\dagger a_i + (|q|^2 - 1) \sum_{k=i+1}^{n} a_k^\dagger a_k
\end{aligned} \tag{7}
$$

To sum up, when the deformation parameter q is a complex number, the q-deformed multimode oscillator algebra becomes

$$
\begin{aligned}
a_i a_j &= q^{-1} a_j a_i \qquad (i < j)\\
a_i^\dagger a_j^\dagger &= q^* a_j^\dagger a_i^\dagger \qquad (i < j)\\
a_i a_j^\dagger &= q a_j^\dagger a_i \qquad (i < j)\\
a_i a_j^\dagger &= q^* a_j^\dagger a_i \qquad (i > j)\\
a_i a_i^\dagger &= 1 + |q|^2 a_i^\dagger a_i + (|q|^2 - 1) \sum_{k=i+1}^{n}
\end{aligned}
\tag{8}
$$

This algebra is covariant under the two-parameter quantum group $gl_{q,q^*}(n)$. So we will refer to this algebra as the $gl_{q,q^*}(n)$-covariant multimode oscillator algebra.

Having obtained the algebra, we can reverse the process and use the algebra itself, together with the definition of the vacuum state,

$$a_i|0\rangle = 0 \qquad \text{for all} \quad i \tag{9}$$

to define the new Fock space. The Fock states are obtained by applying the creation operators $a_i^\dagger$ to the vacuum state $|0\rangle$.

The $gl_{q,q^*}(n)$-covariant oscillator algebra and the definition of the vacuum state then allow us to calculate any matrix element of any polynomial in the a_i. We next construct the number operators N_i for the above system,

$$N_i = \sum_{k=1}^{\infty} \frac{(1 - |q|^2)^k}{(1 - |q|^{2k})} (a_i^\dagger)^k a_i^k |q|^{-2k\Sigma_{i<l} N_l} \tag{10}$$

which satisfies the usual commutation relations

$$
\begin{aligned}
[N_i, a_j^\dagger] &= \delta_{ij} a_j^\dagger\\
[N_i, a_j] &= -\delta_{ij} a_j\\
[N_i, N_j] &= 0
\end{aligned}
\tag{11}
$$

Let us introduce the Fock space basis for the number operators N_1, N_2, $\ldots$, N_n satisfying

$$N_i|n_1, \ldots, n_n\rangle = n_i|n_j, \ldots, n_n\rangle \qquad (n_1, n_2, \ldots, n_n = 0, 1, 2, \ldots) \tag{12}$$

Then we have the representation

$$
\begin{aligned}
a_i|n_1, \ldots, n_n\rangle &= q\Sigma_{k=i+1}^{n} n_k \sqrt{[n_i]}|n_1, \ldots, n_i - 1, \ldots, n_n\rangle\\
a_i^\dagger|n_1, \ldots, n_n\rangle &= q\Sigma_{k=i+1}^{n} n_k \sqrt{[n_i + 1]}|n_1, \ldots, n_i + 1, \ldots, n_n\rangle
\end{aligned}
\tag{13}
$$

where $[x]$ is defined as

$$[x] = \frac{|q|^{2x} - 1}{|q|^2 - 1}$$

From the existence of the ground state $|0, 0, \ldots, 0\rangle$, the state $|n_1, \ldots, n_n\rangle$ is obtained by applying the creation operators to the ground state $|0, 0, \ldots, 0\rangle$

$$|n_1, \ldots, n_n\rangle = \frac{a_n^{\dagger n_n} \cdots a_1^{\dagger n_1}}{\sqrt{[n_1]! \cdots [n_n]!}} |0, 0, \ldots, 0\rangle \tag{14}$$

In what follows, we study the statistics of the many-particle state which is invariant under the quantum group $gl_{q,q^*}(n)$. Let N be the number of such particles. Then the N-particle state can be obtained from the tensor product of the single-particle state:

$$|i_1, \ldots, i_N\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle \tag{15}$$

where $i_1, \ldots i_N$ take one value among $\{1, 2, \ldots, n\}$ and the single-particle state is defined by $|i_k\rangle = a_{i_k}^\dagger|0\rangle$.

Consider the case that k appears n_k times in the set $\{i_1, \ldots, i_N\}$. Then we have

$$n_1 + n_2 + \cdots + n_n = \sum_{k=1}^{n} n_k = N \tag{16}$$

In an analogous way to ordinary symmetric or antisymmetric states in quantum statistical physics, one can define q-symmetric states by the action of a string of creation operators on the vacuum state as follows:

$$|i_1, \ldots, i_N\rangle_q = \sqrt{\frac{[n_1]! \cdots [n_n]!}{[N]!}} \sum_{\sigma \in \mathrm{Perm}} \mathrm{sgn}_q(\sigma) |i_{\sigma(1)} \cdots i_{\sigma(N)}\rangle \tag{17}$$

where

$$\mathrm{sgn}_q(\sigma) = q^{R(i_1 \cdots i_N)} q^{R(\sigma(1) \cdots (N))}$$

$$R(i_1, \ldots, i_N) = \sum_{k=1}^{N} \sum_{l=k+1}^{N} R(i_k, i_l)$$

and

$$R(i, j) = \begin{cases} 1 & \text{if} \quad i > j \\ 0 & \text{if} \quad i \leq j \end{cases}$$

It is worth noting that the above-defined q-symmetric state is $gl_{q,q^*}(n)$-

invariant because these creation operators (associated annihilation operators) satisfy a set of $gl_{q,q^*}(n)$-invariant commutation relations (8). It is clear that once the invariant nature of the commutation relation is specified, the resulting state vector is endowed with the invariance property of the related algebra [in this case $gl_{q,q^*}(n)$-covariant multimode oscillator algebra].

Then the q-symmetric state obeys

$$|\ldots, i_k, i_{k+1}, \ldots\rangle_g = \begin{cases} q^{-1}|\ldots, i_{k+1}, i_k, \ldots\rangle_q & \text{if} \quad i_k < i_{k+1} \\ |\ldots, i_{k+1}, i_k, \ldots\rangle_q & \text{if} \quad i_k = i_{k+1} \\ q|\ldots, i_{k+1}, i_k, \ldots\rangle_q & \text{if} \quad i_k > i_{k+1} \end{cases} \tag{18}$$

The above property can be rewritten by introducing the deformed transition operator $P_{k,k+1}$ obeying

$$P_{k,k+1}|\ldots, i_k, i_{k+l}, \ldots\rangle_q = |\ldots, i_{k+1}, i_k, \ldots\rangle_q \tag{19}$$

This operator satisfies

$$P_{k+1,k}P_{k,k+1} = Id, \qquad \text{so} \quad P_{k+1,k} = P^{-1}_{k,k+1} \tag{20}$$

Then equation (18) can be written as

$$P_{k,k+1}|\ldots, i_k, i_{k+1}, \ldots\rangle_q = q^{-\epsilon(i_k,i_{k+1})}|\ldots, i_{k+1}, i_k, \ldots\rangle_q \tag{21}$$

where $\epsilon(i, j)$ is defined as

$$\epsilon(i, j) = \begin{cases} 1 & \text{if} \quad i > j \\ 0 & \text{if} \quad i = j \\ -1 & \text{if} \quad i < j \end{cases}$$

The relation (21) goes over to the symmetric relation for ordinary bosons when the deformation parameter q goes to 1. If we define the fundamental q-symmetric state $|q\rangle$ as

$$|q\rangle = |i_1, i_2, \ldots, i_N\rangle_q$$

with $i_1 \leq i_2 \leq \ldots \leq i_N$, we have

$$||q\rangle|^2 = 1$$

In deriving the above relation we used the identity

$$\sum_{\sigma \in \mathrm{Perm}} |q|^{2R(\sigma(\perp),\ldots,\sigma(N))} = \frac{[N]!}{[n_1]! \cdots [n_n]!} \tag{22}$$

In this paper I have extended the $gl_q(n)$-covariant oscillator algebra to the complex deformation parameter case by using the R-matrix formalism.

The q-symmetric states generalizing the symmetric (bosonic) states and antisymmetric (fermionic) states are obtained by using $gl_{q,q^*}(n)$-covariant oscillators and are shown to be orthonormal. I think that the q-symmetric states will be important when we consider the new statistical field theory (q-deformed statistical field theory) generalizing the ordinary one.

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